

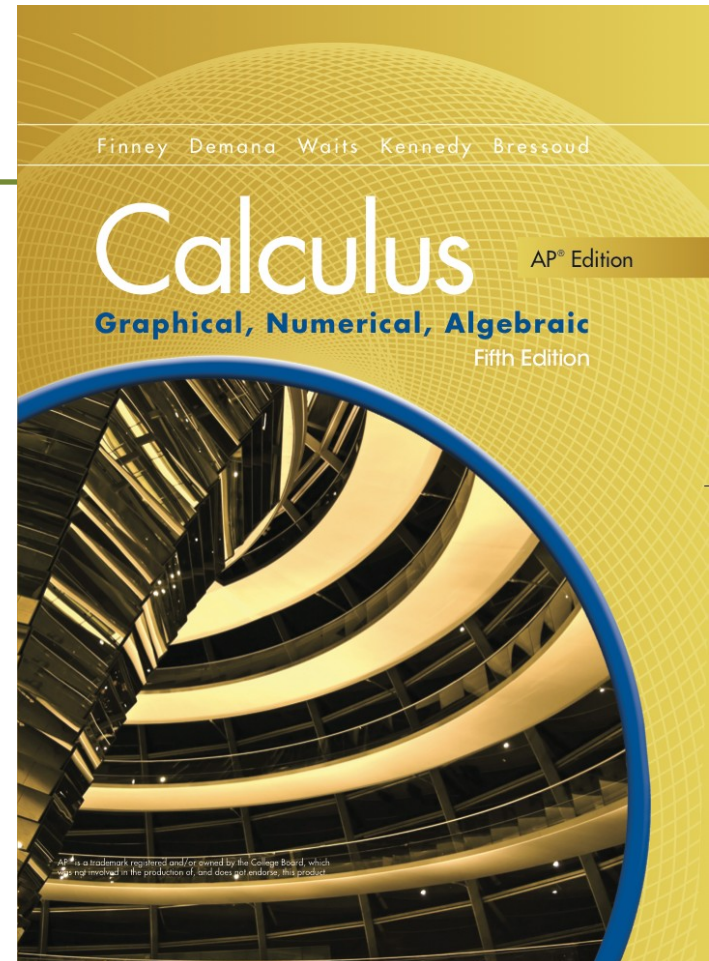
Chapter 9

Sequences, L'Hopital's Rule, and Improper Integrals



Section 9.4

Improper Integrals



Quick Review

Evaluate the integral.

1. $\int_0^3 \frac{dx}{x+3}$

2. $\int_1^4 \frac{2x dx}{x^2 + 1}$

3. $\int \frac{dx}{1+x^2}$

4. $\int \frac{dx}{x^2}$

Find the domain of the function.

5. $f(x) = \frac{1}{\sqrt{4-x^2}}$

6. $g(x) = \frac{1}{\sqrt{x-4}}$

Quick Review

7. Confirm the inequality: $\left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2}, \quad -\infty < x < \infty$

8. Show that $f(x) = 4e^x + 5$ and $g(x) = 5e^x - 3$ grow at the same rate as $x \rightarrow \infty$.

Quick Review Solutions

Evaluate the integral.

1. $\int_0^3 \frac{dx}{x+3}$ $\ln 2$

2. $\int_1^1 \frac{2xdx}{x^2+1}$ 0

3. $\int \frac{dx}{1+x^2}$ $\tan^{-1}x + C$

4. $\int \frac{dx}{x^2}$ $-x^{-1} + C$

Find the domain of the function.

5. $f(x) = \frac{1}{\sqrt{4-x^2}}$ $(-2, 2)$

6. $g(x) = \frac{1}{\sqrt{x-4}}$ $(4, \infty)$

Quick Review Solutions

7. Confirm the inequality: $\left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2}, \quad -\infty < x < \infty$

Because $-1 \leq \cos x \leq 1$ for all x

8. Show that $f(x) = 4e^x + 5$ and $g(x) = 5e^x - 3$ grow at the same rate as $x \rightarrow \infty$.

$$\lim_{x \rightarrow \infty} \frac{4e^x + 5}{5e^x - 3} = \frac{4}{5}$$

What you'll learn about

- Improper integrals
- Horizontally improper integrals (infinite limits of integration)
- Vertically improper integrals (intervals with points of infinite discontinuity)
- Comparison tests for convergence and divergence of improper integrals

...and why

The techniques of this section allow us to extend integration techniques to cases where the interval of integration $[a,b]$ is not finite or where integrands are not continuous.

Improper Integrals with Infinite Integration Limits

Integrals with infinite limits of integration are improper integrals.

1. If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$
2. If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$, where c is any real number.

Example Evaluating an Improper Integral on $[1, \infty)$

Does the improper integral $\int_1^{\infty} \frac{dx}{x}$ converge or diverge?

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \ln x \Big|_1^b \\ &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) = \infty\end{aligned}$$

Thus, the integral diverges.

Example Using L'Hôpital's Rule with Improper Integrals

Evaluate $\int_1^{\infty} x e^{-x} dx$

$$\int_1^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b x e^{-x} dx$$

Use integration by parts to evaluate the definite integral:

$$u = x \quad dv = e^{-x} dx$$

$$du = dx \quad v = -e^{-x}$$

$$\begin{aligned} \int_1^b x e^{-x} dx &= \left[-x e^{-x} \right]_1^b + \int_1^b e^{-x} dx \\ &= \left[-x e^{-x} - e^{-x} \right]_1^b \\ &= \left[-e^{-x}(x+1) \right]_1^b \\ &= -e^{-b}(b+1) + 2e^{-1} \end{aligned}$$

$$\lim_{b \rightarrow \infty} \left[-e^{-b}(b+1) + 2e^{-1} \right] = \lim_{b \rightarrow \infty} \left(\frac{-1}{e^b} + \frac{2}{e} \right) = \frac{2}{e}.$$

Example Evaluating an Integral on $(-\infty, \infty)$

Evaluate $\int_{-\infty}^{\infty} \frac{2dx}{1+x^2}$.

According to the definition $\int_{-\infty}^{\infty} \frac{2dx}{1+x^2} = \int_{-\infty}^0 \frac{2dx}{1+x^2} + \int_0^{\infty} \frac{2dx}{1+x^2}$.

Evaluate each improper integral:

$$\begin{aligned}\int_{-\infty}^0 \frac{2dx}{1+x^2} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{2dx}{1+x^2} \\ &= 2 \lim_{a \rightarrow -\infty} \left[\tan^{-1} x \right]_a^0 \\ &= 2 \lim_{a \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 2 \left(\frac{\pi}{2} \right) = \pi\end{aligned}$$

$$\begin{aligned}\int_0^{\infty} \frac{2dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{2dx}{1+x^2} \\ &= 2 \lim_{b \rightarrow \infty} \left[\tan^{-1} x \right]_0^b \\ &= 2 \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = 2 \left(\frac{\pi}{2} \right) = \pi\end{aligned}$$

$$\text{So, } \int_{-\infty}^{\infty} \frac{2dx}{1+x^2} = \pi + \pi = 2\pi$$

Improper Integrals with Infinite Discontinuities

Integrals of functions that become infinite at a point within the interval of integration are improper integrals.

1. If $f(x)$ is continuous on $(a, b]$, then $\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$

2. If $f(x)$ is continuous on $[a, b)$, then $\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$

3. If $f(x)$ is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Example Infinite Discontinuity at an Interior Point

Evaluate $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

The integrand has a vertical asymptote at $x=1$ and is continuous on

$[0,1)$ and $(1,2]$. By definition $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

Evaluate each improper integral:

$$\begin{aligned} \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} &= \lim_{c \rightarrow 1^-} \int_0^c \frac{dx}{(x-1)^{\frac{2}{3}}} \\ &= \lim_{c \rightarrow 1^-} 3(x-1)^{\frac{1}{3}} \Big|_0^c \\ &= \lim_{c \rightarrow 1^-} \left[3(c-1)^{\frac{1}{3}} + 3 \right] = 3. \end{aligned}$$

$$\begin{aligned} \int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}}} &= \lim_{c \rightarrow 1^+} \int_c^2 \frac{dx}{(x-1)^{\frac{2}{3}}} \\ &= \lim_{c \rightarrow 1^+} 3(x-1)^{\frac{1}{3}} \Big|_c^2 \\ &= \lim_{c \rightarrow 1^+} \left[3 - 3(c-1)^{\frac{1}{3}} \right] = 3. \end{aligned}$$

Therefore $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3 = 6$.

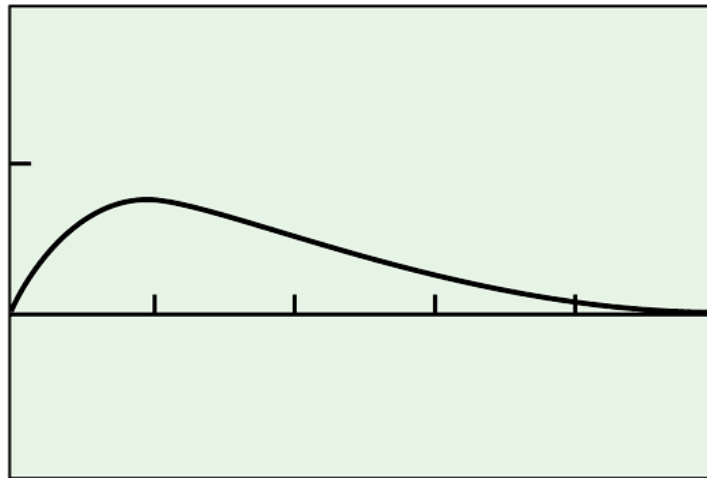
Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^{\infty} f(x) dx$ converges if $\int_a^{\infty} g(x) dx$ converges.
2. $\int_a^{\infty} g(x) dx$ diverges if $\int_a^{\infty} f(x) dx$ diverges.

Example Finding the Volume of an Infinite Solid

Find the volume of the solid obtained by revolving the curve $y = xe^{-x}$, $0 \leq x < \infty$ about the x -axis.



$[0, 5]$ by $[-0.5, 1]$

The graph above shows a portion of the region to be revolved about the x -axis.

Example Finding the Volume of an Infinite Solid

The area of a typical cross section of the solid is: $\pi (\text{radius})^2 = \pi y^2 = \pi x^2 e^{2x}$.

$$\text{Therefore } V = \pi \int_0^{\infty} x^2 e^{2x} dx = \pi \lim_{b \rightarrow \infty} \int_0^b x^2 e^{2x} dx$$

Integrating by parts twice we obtain the following.

$$\begin{aligned} \int x^2 e^{2x} dx &= -\frac{x^2}{2} e^{2x} + \int x e^{2x} dx \\ &= -\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{2} \int e^{2x} dx \\ &= -\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C \\ &= -\frac{2x^2 + 2x + 1}{4e^{2x}} + C \end{aligned}$$

$$\begin{aligned} \text{Thus, } V &= \pi \lim_{b \rightarrow \infty} \left[-\frac{2x^2 + 2x + 1}{4e^{2x}} \right]_0^b \\ &= \pi \lim_{b \rightarrow \infty} \left[-\frac{2b^2 + 2b + 1}{4e^{2b}} + \frac{1}{4} \right] = \frac{\pi}{4}. \end{aligned}$$

Quick Quiz Sections 9.3 and 9.4

You may use a graphing calculator to solve the following problems.

1. Which of the following functions grows faster than x^2 as $x \rightarrow \infty$?

(A) e^x

(B) $\ln x$

(C) $7x+10$

(D) $2x^2 - 3x$

(E) $0.1x^3$

Quick Quiz Sections 9.3 and 9.4

You may use a graphing calculator to solve the following problems.

1. Which of the following functions grows faster than x^2 as $x \rightarrow \infty$?

(A) e^{-x}

(B) $\ln x$

(C) $7x+10$

(D) $2x^2 - 3x$

(E) $0.1x^3$

Quick Quiz Sections 9.3 and 9.4

2. Find all values of p for which the integral converges $\int \frac{dx}{x^{p+1}}$.

(A) $p < -1$

(B) $p < 0$

(C) $p > 0$

(D) $p > 1$

(E) diverges for all p

Quick Quiz Sections 9.3 and 9.4

2. Find all values of p for which the integral converges $\int_1^{\infty} \frac{dx}{x^{p+1}}$.

(A) $p < -1$

(B) $p < 0$

(C) $p > 0$

(D) $p > 1$

(E) diverges for all p

Quick Quiz Sections 9.3 and 9.4

3. Find all the values of p for which the integral converges $\int_0^1 \frac{dx}{x^{p+1}}$.

(A) $p < -1$

(B) $p < 0$

(C) $p > 0$

(D) $p > 1$

(E) diverges for all p

Quick Quiz Sections 9.3 and 9.4

3. Find all the values of p for which the integral converges $\int_0^1 \frac{dx}{x^{p+1}}$.

(A) $p < -1$

(B) $p < 0$

(C) $p > 0$

(D) $p > 1$

(E) diverges for all p

Chapter Test

1. Find the first four terms and the fortieth term of the given sequence:

$$a_n = (-1)^n \frac{n+1}{n+3} \text{ for all } n \geq 1$$

2. The sequence $1/2, -2, 8, -32, \dots$ is geometric. Find **(a)** the common ratio, **(b)** the seventh term, and **(c)** an explicit rule for the n th term.

3. Determine the convergence or divergence of the sequence with given n th term. If the sequence converges, find its limit. $a_n = (-1)^n \frac{3n-1}{n+2}$

Chapter Test

Find the limit

$$4. \lim_{t \rightarrow 0} \frac{t - \ln(1 + 2t)}{t^2}$$

$$5. \lim_{x \rightarrow \infty} x^{\frac{1}{x}}$$

Determine whether f grows faster than, slower than, or at the same rate as g as $x \rightarrow \infty$.

$$6. f(x) = x, g(x) = x + \frac{1}{x}$$

$$7. f(x) = x, g(x) = \tan^{-1} x$$

Chapter Test

8. Given $f(x) = \frac{2^{\sin x} - 1}{e^x - 1}$,

(a) show that f has a removable discontinuity at $x = 0$.

(b) define f at $x = 0$ so that it is continuous.

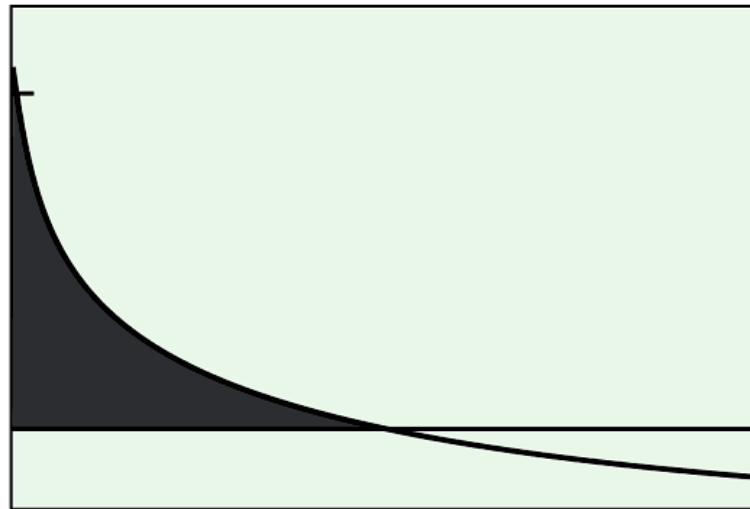
Evaluate the improper integral or state that it diverges.

9. $\int_1^{\infty} \frac{dx}{x^2}$

10. $\int_1^{\infty} \frac{dx}{e^x + e^{-x}}$

Chapter Test

11. The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x -axis to generate a solid. Find the volume of the solid.



$[0, 2]$ by $[-1, 5]$

Chapter Test Solutions

1. Find the first four terms and the fortieth term of the given sequence:

$$a_n = (-1)^n \frac{n+1}{n+3} \text{ for all } n \geq 1 \quad -1/2, 3/5, -2/3, 5/7; a_{40} = 41/43$$

2. The sequence $1/2, -2, 8, -32, \dots$ is geometric. Find **(a)** the common ratio, **(b)** the seventh term, and **(c)** an explicit rule for the n th term.

(a) -4 **(b)** 2048 **(c)** $a_n = (-1)^{n-1} (2^{2n-3})$

3. Determine the convergence or divergence of the sequence with given

n th term. If the sequence converges, find its limit. $a_n = (-1)^n \frac{3n-1}{n+2}$

diverges

Chapter Test Solutions

Find the limit

4. $\lim_{t \rightarrow 0} \frac{t - \ln(1 + 2t)}{t^2}$ does not exist

5. $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$ 1

Determine whether f grows faster than, slower than, or at the same rate as g as $x \rightarrow \infty$.

6. $f(x) = x$, $g(x) = x + \frac{1}{x}$ same rate

7. $f(x) = x$, $g(x) = \tan^{-1} x$ faster

Chapter Test Solutions

8. Given $f(x) = \frac{2^{\sin x} - 1}{e^x - 1}$,

(a) show that f has a removable discontinuity at $x=0$. $\lim_{x \rightarrow 0} f(x) = \ln 2$

(b) define f at $x=0$ so that it is continuous. define $f(0) = \ln 2$

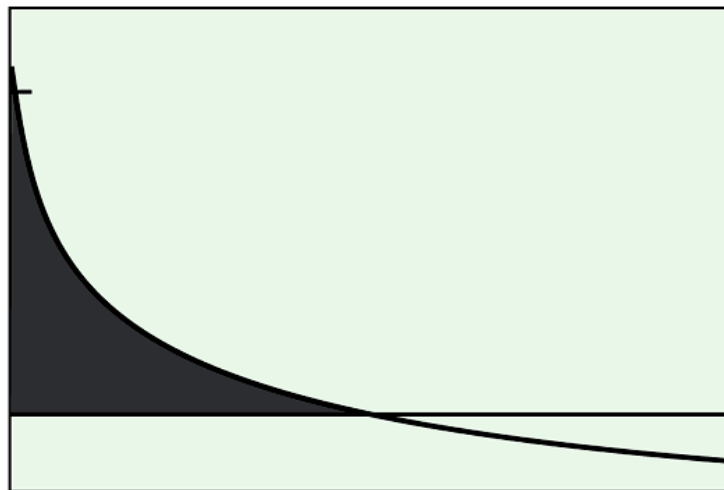
Evaluate the improper integral or state that it diverges.

9. $\int_1^{\infty} \frac{dx}{x^2}$ 2

10. $\int_1^{\infty} \frac{dx}{e^x + e^{-x}}$ $\pi/2 - \tan^{-1}(e)$

Chapter Test Solution

11. The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x -axis to generate a solid. Find the volume of the solid. 2π



$[0, 2]$ by $[-1, 5]$